**Table 1 Convergence study of skew shell panel ($\bar{\alpha}=45°$, $h/a=0.01$, $H/h=10.16$, $\ell=a=1$, $\nu=0.3$)**

| No of harmonics | Mode | $\omega(\rho/E)^{1/2}$ [a] No of strips 2 | 3 | 4 |
|---|---|---|---|---|
| 4 | 1 | 0.079 | 0.078 | 0.077 |
| 4 | 2 | 0.107 | 0.100 | 0.099 |
| 4 | 3 | 0.137 | 0.117 | 0.116 |
| 4 | 4 | 0.145 | 0.137 | 0.134 |
| 6 | 1 | 0.079 | 0.077 | 0.077 |
| 6 | 2 | 0.103 | 0.097 | 0.096 |
| 6 | 3 | 0.136 | 0.115 | 0.114 |
| 6 | 4 | 0.143 | 0.134 | 0.132 |

[a] Has a dimension of 1/length.

$$-E_{12}k_n^2[r_i(r_i-1)$$
$$+r_j(r_j-1)]F(r_i+r_j-1\}(\ell/2)\ \delta_{mn}$$
$$+\{2E_{13}r_ir_j[k_n(r_i-1)I_{m,n}$$
$$+k_m(r_j-1)I_{n,m}]F(r_i+r_j-2)$$
$$+2E_{23}k_mk_n(k_mr_jI_{m,n}$$
$$+k_nr_iI_{n,m})F(r_i+r_j)\}\rangle(h^3/12)\sin\alpha$$

where $F(m)=b^m/m$; $k_m=m\pi/\ell$; $\delta_{mn}$ is the Kronecker delta; and

$$I_{m,n}=\int_0^\ell \sin(\frac{m\pi y}{\ell})\cos(\frac{n\pi y}{\ell})dy$$

Then the stiffness matrix $[K]$ consists of $N\times N$ submatrices, typical of which is $[K_{mn}]=[T]^T[\hat{K}]_{mn}[T]$.

Similarly, combining Eqs. (5) and (6), the mass matrix $[M]$ is given by diagonal submatrices such as $[M]_{11}$, $[M]_{22}\ldots$, $[M]_{NN}$, in which $[M]_{nn}=[Tw]^T[\hat{M}]_{nn}\ [Tw]$ and $(\hat{M}_{ij})_{nn}=(\ell/2)\ (b^{r_i+r_j+1})/(r_i+r_j+1)$. After assembling the stiffness and mass matrices, the geometrical boundary conditions in the $x$ direction are introduced, and then this eigenvalue problem is solved.

## Numerical Work and Disucssion

The numerical work has been done for an isotropic panel supported on shear diaphragms on all edges. Convergence of frequency has been studied by taking different numbers of harmonics and also increasing the number of strips (see Table 1), The convergence of the solution is good. Further calculations have been done using four strips and six harmonics. The frequencies for the particular cases (viz. curved rectangular panel and flat skew plate) were compared (not reported here) with those of Sewall[1] and Durvasula,[7] respectively, and they were found to agree well.

The influence of rise and skew angle on the fundamental frequency parameter $[\omega(\rho/E)^{1/2}]$, which has the dimension of 1/length, is given beneath each nodal pattern.

As the rise of the panel is increased, the membrance action comes into play, and this is evidenced by the rapid increase in the frequency of the nodeless mode (see Fig. 3). When $H/h>5$, the lowest mode for all of the skew angles considered has a single nodal line. For this mode shape, the energy will be equally due to bending and stretching, whereas for higher modes with more than one vertically oriented node the energy will be predominently due to bending. For the skewed panels at $H/h\geq5$, explicit distinction between membrane and bending effects is obscured by the disappearance of the nodeless modes and the existence of highly curved modes. By choosing proper functions in the $y$ direction, shells with different types of boundary conditions can be analyzed.

## Appendix

$$[Tu]=[Tv]=\begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ -3/B^2 & -2/B & 3/B^2 & -1/B \\ 2/B^3 & 1/B^2 & -2/B^3 & 1/B^2 \end{bmatrix}$$

$$[Tw]=\begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ -10/B^3 & -6/B^2 & -1.5/B & 10/B^3 & -4/B^2 & 0.5/B \\ 15/B^4 & 8/B^3 & 1.5/B^2 & -15/B^4 & 7/B^3 & -1/B^2 \\ -6/B^5 & -3/B^4 & -0.5/B^3 & 6/B^5 & -3/B^4 & 0.5/B^3 \end{bmatrix}$$

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# Consistent Mass Matrix in Fluid Sloshing Problems

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SEVERAL recent publications have presented a technique for the solution of problems involving the free-surface vibration of fluids in a container by a finite element method.[1-4] In these works Lagrange's equation is used to formulate a standard matrix vibration equation using the displacement formulation common to most finite element structural analysis.

In this present Note it is shown that a derivation of the mass matrix for such sloshing problems based on lumping the effective vibration mass at various nodes gives much inferior results in comparison to a consistent mass matrix. The consistent mass matrix is of the same bandwidth as the stiffness matrix whereas the lumped mass approach has only the diagonal terms nonzero. Thus, a consistent mass formulation implies a time penalty on computations.

With structural and elasticity vibration problems, there is much evidence[5,6] (and many more) that there is only a

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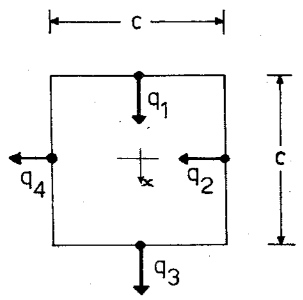


**Fig. 1 Square element of side C with four degrees of freedom.**

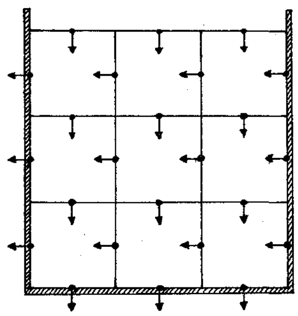

**Fig. 2 3 by 3 grid of elements with 24 degrees of freedom representing fluid in tank of aspect ratio one.**

marginal difference in accuracy between the results obtained for lower modes using the two types of mass matrix and consequently for reasons mentioned above the lumped mass matrix is favored for such problems. However opinion is divided, as the work of Archer[7] indicates that, for certain structural problems involving beam elements, a consistent mass matrix gives order-of-magnitude improvement over a lumped mass matrix.

It can be shown[2,4] that the motion of an incompressible fluid in a rigid container in two dimensions is governed by a simple vibration equation. In this equation the mass matrix $[M]$ and stiffness matrix $[K]$ are assembled by the direct stiffness method from element matrices $[M^e]$ and $[K^e]$.

A simple element in two dimensions consisting of a square of side $c$ shall be adopted to illustrate the two differing approaches to the mass matrix, Fig. 1. Four nodes are located at the mid point of each side. Each node has a single degree of freedom $q_i$, thus two nodes cater for horizontal motion and two for vertical motion. Thus, since horizontal and vertical motion are uncoupled, it is only necessary to look at one pair, say for vertical motion, nodes 1 and 3. These nodes must accommodate the mass of the element and in essence, the lumped mass concept states that the effective vibration mass would be divided evenly between them. The superscript $e$ refers to an element quantity.

$$[M^e] = \frac{\rho c^2}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

**Table 1 λ = wavelength, *h* = height of tank**

| Square of frequency | Consistent mass | Lumped mass | Lamb[8] |
|---|---|---|---|
| $\omega_1{}^2$ | 1.0434 $g/_c$ | 0.8888 $g/_c$ | 1.0432 $g/_c$ |
| $\omega_2{}^2$ | 2.0000 $g/_c$ | 1.3090 $g/_c$ | 2.0944 $g/_c$ |

The consistent mass matrix approach adopts a shape function for element mass distribution compatible with that for displacement. Consequently, with two nodes to describe vertical displacement, this shape function will be linear, i.e.

$$N_1 = \tfrac{1}{2} - x/c, \; N_3 = x/c + \tfrac{1}{2} \tag{2}$$

These give the mass matrix for vertical motion as

$$\int_e \rho \begin{bmatrix} N_1N_1 & N_1N_3 \\ N_3N_1 & N_3N_3 \end{bmatrix} dV = \frac{\rho c^2}{2} \begin{bmatrix} \tfrac{2}{3} & \tfrac{1}{3} \\ \tfrac{1}{3} & \tfrac{2}{3} \end{bmatrix} \tag{3}$$

and for the whole element

$$[M^e] = \frac{\rho c^2}{2} \begin{bmatrix} \tfrac{2}{3} & \tfrac{1}{3} & 0 & 0 \\ \tfrac{1}{3} & \tfrac{2}{3} & 0 & 0 \\ 0 & 0 & \tfrac{2}{3} & \tfrac{1}{3} \\ 0 & 0 & \tfrac{1}{3} & \tfrac{2}{3} \end{bmatrix} \tag{4}$$

In a U tube manometer the change in potential is the weight per unit length times the change in height, and is independent of the shape of the tube. Terms in the stiffness matrix $[K^e]$ can be derived by use of the unit displacement theorem of structural analysis whereby $K_{ij}$ is the force at $i$ due to unit displacement at $j$, all other displacements being zero. Thus, in this model of fluid subject to surface motion, the force required for unit displacement is the weight per unit depth which is $\rho gc$ and it only affects the single surface node of free-surface elements.

$$[K^e] = gc\rho \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

As an example of the application of this mathematical model of fluid sloshing within a rigid container, a 3 by 3 grid of elements is selected; Fig. 2. This has 24 degrees of freedom but this is reduced to 15 since wall nodes are fived. The overall mass and stiffness matrices are assembled in the usual way from the elemental ones. These matrices can be reduced considerably by use of the fluid incompressibility constraint conditions for each element, i.e., for element in Fig. 1

$$q_1 + q_2 - q_3 - q_4 = 0 \tag{6}$$

The reduced mass and stiffness matrices are now 6 by 6. By appropriate Cholesky decomposition a standard eigenvalue equation may be obtained form which the natural frequencies of surface motion are obtained. Note that, because there are only three surface nodes only two mode shapes are possible, one a half wave of wavelength $6c$, the other a full wave of wavelength $3c$. For this reason only two nonzero frequencies are produced in the solution of the eigenvalue equation. It can be argued that the zero frequency modes are associated with internal motions in the system which do not produce surface motion and it is possible to reduce the size of the eigenvalue equation by incorporating an influence matrix between the zero modes and the nonzero modes.

The table below shows the square of these two non-frequencies for both the lumped mass and consistent mass approaches. These are compared with Lamb's[8] analytical solution to the same problem

$$\omega^2 = \frac{2\pi g}{\lambda} \tanh \left[ \frac{2\pi g}{\lambda} \right] \tag{7}$$

These figures provide a dramatic illustration of the improvement of accuracy provided by the use of the consistent mass matric approach to such problems. Even with a 7 by 7 grid the lumped mass matrix approach was still 3.2% different from Lamb's value of $\omega_I{}^2$.

In conclusion, it should be pointed out that the application of the fluid incompressibility constraint conditions renders the reduced form of the mass and stiffness matrices fully populated in both cases and consequently the arguments which favoured the lumped mass matrix approach for structural problems disappear and from the above table it is clear that only a consistent mass matrix should be used.

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# Stability Derivatives for Bodies of Revolution at Subsonic Speeds

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## I. Introduction

RECENT interest in slender wing/body and airship design in subsonic flight demands improved techniques for their static and dynamic stability prediction. For the steady lifting case, Refs.1-4 have derived higher approximations based on the linearized small-perturbation potential equation, i.e., Eq. (4). More exact theories were given by Hess and Smith[5] in a numerical approach and by Revell[6] in a matched asymptotic expansion procedure. For the oscillatory case, Revell's work appears to be the only available one, suitable for the dynamic-stability calculation.

The purpose of this Note is twofold. First, the present method employs a comparatively simpler procedure than Revell's in extending the earlier work[1-4] to the unsteady case. The result obtained can be expressed simply in algebraic form, and therefore it can be adopted easily for practical design calculations. Second, the present investigation serves as a numerical check on the limitation of the subsonic linearized equation in comparison to the second-order equation derived by Revell.

In Revell's formulation, the oscillatory-flow equations [Eqs. (27) and (54) of Ref. 6] are both inhomogeneous equations that account for the coupling effects with the steady mean flow. In the present formulation, the governing equation used is a linear homogeneous one, in which the mean flow influence is absent.§ Nevertheless, the mean flow influence is recovered partly in the present solution by asymptotic expansion of the near-flowfield through the flow tangency condition, Eq. (11), and the pressure formula Eqs. (14) and (15). In this way, the effects of freestream Mach number $M$, body shape $R(x)$, and body thickness $\epsilon$ can be accounted for in the stability derivative calculation. We remark that the present procedure is essentially the same as the quasislender body theory for supersonic flow given by Platzer and Hoffman[7] and is considered to be its subsonic counterpart.

## II. Problem Formulation

Consider a rigid, pointed body of revolution which is exposed to a steady uniform subsonic flow and which performs harmonic, small-amplitude pitching oscillations around its zero angle of attack position. A body-fixed cylindrical coordinate system, as shown in Fig. 1, is used to describe the problem.¶ The body is assumed to be smooth and sufficiently slender so that the small-perturbation concept can be applied. Let $\delta_0$ represent the amplitude of oscillation and $k$ the reduced frequency. Following Revell,[8] a body-fixed perturbation potential $\Phi(x,r,\theta,t)$ can be related to the general velocity potential $\Omega(x,r,\theta,t)$ as follows:

$$\Omega(x,r,\theta,t) = (x-a)\cos\delta + r\sin\delta\cos\theta + \Phi(x,r,\theta,t) \tag{1}$$

where

$$\delta = \delta_0 e^{ikt} \tag{2}$$

is the pitch angle. The velocity components in the $x,r,\theta$ directions then are given by

$$u = \Omega_x = \cos\delta + \Phi_x \tag{3a}$$

$$v = \Omega_r = \sin\delta\cos\theta + \Phi_r \tag{3b}$$

$$w = (1/r)\Omega_\theta = -\sin\delta\sin\theta + (1/r)\Phi_\theta \tag{3c}$$

The first-order equation governing unsteady subsonic flow is

$$(1-M^2)\Phi_{xx} + \Phi_{rr} + (1/r)\Phi_r + (1/r^2)\Phi_{\theta\theta} - 2M^2\Phi_{xt} - M^2\Phi_{tt} = 0 \tag{4}$$

For a discussion of the parametric restrictions imposed on this equation, see Ref. 10. Assuming harmonic time dependence, the perturbation potential can be written as

$$\Phi(x,r,\theta,t) = \phi(x,r) + \varphi(x,r,\theta)e^{ikt} \tag{5}$$

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§The authors are grateful to J. D. Revell for stressing this point.

¶All variables are nondimensional, with distances referred to body length, velocities to freestream speed, and time to body length divided by freestream speed.